1. (4.4 in NC) Express the Hadamard gate $H$ as a product of $R_x$ and $R_z$ rotations and $e^{i\phi}$ for some $\phi$.

2. (NC problems 4.13, 4.14) Prove the following
   (a) $H X H = Z$
   (b) $H Y H = -Y$
   (c) $H Z H = X$
   (d) $H T H = R_x(\pi/4)$ up to a global phase.

3. Exercise 4.20 in NC. We have described how the CNOT behaves with respect to the computational basis where the state of the control qubit does not change. However, if we work in a different basis the control qubit does change. Show that

   Introducing basis states $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$, use this circuit identity to show the effect of a CNOT with the first qubit as control and second qubit as target to be

   $|+\rangle |+\rangle \rightarrow |+\rangle |+\rangle$ \hspace{1cm} (1)
   $|-\rangle |+\rangle \rightarrow |-\rangle |+\rangle$ \hspace{1cm} (2)
   $|+\rangle |-\rangle \rightarrow |-\rangle |-\rangle$ \hspace{1cm} (3)
   $|-\rangle |-\rangle \rightarrow |+\rangle |-\rangle$ \hspace{1cm} (4)
4. Exercise 4.18 in NC. (Building CNOT from controlled-Z gates) Show that the following are equivalent:

![Diagram](image)

5. Exercise 4.25 in NC. (Fredkin gate construction) The Fredkin (controlled swap) gate performs the following operation on 3 qubits

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

(a) Give a quantum circuit that uses three Toffoli gates to construct the Fredkin gate (Hint: think of the swap gate construction - you can control each gate one at a time).

(b) Show that the first and last Toffoli gates can be replaced by CNOT gates.

(c) Now replace the middle Toffoli gate with the circuit given in class for the $C^2(U)$ gate to obtain a Fredkin gate construction using only six two-qubit gates.

(d) Can you come up with a simpler construction using only five two-qubit gates?