When preparing your written solutions, please show all steps involved in obtaining your answer. This will make it easier to assign partial credit (for partially correct answers) and will also help me (or the grader) in helping you to see where (if anywhere) you made a mistake.

1. Find the density matrices for the whole system and both qubits of

(a) \( |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \)
(b) \( |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \)

Ans:

(a) The density matrix for the whole system:

\[
\rho_{\Psi^-} = |\Psi^-\rangle \langle \Psi^-| = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}
\]

The density matrix for the first qubit:

\[
\rho_{1\Psi^-} = tr_2(\rho_{\Psi^-}) = \frac{1}{2} (|0\rangle \langle 0| + |1\rangle \langle 1|) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}
\]

The density matrix for the second qubit:

\[
\rho_{2\Psi^-} = tr_1(\rho_{\Psi^-}) = \frac{1}{2} (|0\rangle \langle 0| + |1\rangle \langle 1|) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}
\]
(b) The density matrix for the whole system:

\[
\rho_{\Phi^+} = |\Phi^+ \rangle \langle \Phi^+ |
\]

\[
= \frac{1}{2} \left( |01 \rangle \langle 01 | + |10 \rangle \langle 01 | + |01 \rangle \langle 10 | + |10 \rangle \langle 10 | \right)
\]

\[
= \frac{1}{2} \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

The density matrix for the first qubit:

\[
\rho^1_{\Phi^+} = tr_2(\rho_{\Phi^+})
\]

\[
= \frac{1}{2} (|0 \rangle \langle 0 | + |1 \rangle \langle 1 |)
\]

\[
= \begin{bmatrix}
\frac{1}{2} & 0 \\
0 & \frac{1}{2}
\end{bmatrix}
\]

The density matrix for the second qubit:

\[
\rho^2_{\Phi^+} = tr_1(\rho_{\Phi^+})
\]

\[
= \frac{1}{2} (|1 \rangle \langle 1 | + |0 \rangle \langle 0 |)
\]

\[
= \begin{bmatrix}
\frac{1}{2} & 0 \\
0 & \frac{1}{2}
\end{bmatrix}
\]
2. We defined a density matrix to be a representation of a probability distribution over a set of states. Show by example that some density matrices have multiple associated probability distributions so that in general the distribution associated with a set of states is not unique.

**Ans:**
We can show that the density matrix for the probability distribution that \(|+\rangle\) is \(\frac{1}{2}\) and \(|-\rangle\) is \(\frac{1}{2}\) is the same as the density matrix for the probability distribution that \(|0\rangle\) is \(\frac{1}{2}\) and \(|1\rangle\) is \(\frac{1}{2}\).

\[
\frac{1}{2}(|+\rangle\langle+| + |-\rangle\langle-|) = \frac{1}{2} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}(|0\rangle\langle0| + |1\rangle\langle1|)
\]
3. (Problem 2.58 in NC) Suppose we prepare a quantum system in an eigenstate $|\psi\rangle$ of some observable $X$, with corresponding eigenvalue $\lambda$. What is the average observed value of $X$, and the standard deviation?

**Ans:**

The average observed value of $X$ is:

$$E(X) = \langle \psi | X | \psi \rangle$$

$$= \langle \psi | \lambda | \psi \rangle$$

$$= \lambda \langle \psi | \psi \rangle$$

$$= \lambda$$

The standard deviation is:

$$\Delta(X) = \sqrt{\langle X^2 \rangle - \langle X \rangle^2}$$

$$= \sqrt{\langle \psi | XX | \psi \rangle - \lambda^2}$$

$$= \sqrt{\lambda \langle \psi | X | \psi \rangle - \lambda^2}$$

$$= \sqrt{\lambda^2 \langle \psi | \psi \rangle - \lambda^2}$$

$$= \sqrt{\lambda^2 - \lambda^2}$$

$$= 0$$
4. Product probability spaces.

(a) Let $p \in \mathbb{R}^d$ be a probability distribution on $[d] = \{1, 2, \ldots, d\}$. Let $q \in \mathbb{R}^e$ be a probability distribution on $[e] = \{1, 2, \ldots, e\}$. Prove that the Kronecker product $p \otimes q$ (which is a vector naturally indexed by the set $[d] \times [e]$) is the associated “product probability distribution” on $[d] \times [e] = \{(i, j) : 1 \leq i \leq d, 1 \leq j \leq e\}$; i.e., it’s the distribution gotten by drawing $i$ from $p$ and $j$ from $q$ independently.

(b) Let $(p_1, |\psi_1\rangle), \ldots, (p_m, |\psi_m\rangle)$ be the mixed state of a $d$-dimensional particle (meaning we have probability $p_i$ of pure state $|\psi_i\rangle \in \mathbb{C}^d$, $i = 1, \ldots, m$). Similarly, let $(q_1, |\phi_1\rangle), \ldots, (q_n, |\phi_n\rangle)$ be the mixed state of a $e$-dimensional particle. Write $\rho \in \mathbb{C}^{d \times d}$ for the density matrix of the first mixed state and $\rho' \in \mathbb{C}^{e \times e}$ for the density matrix of the second. Suppose the particles were created completely separately and independently, but we now decide to view them as a joint $de$-dimensional state. Recalling the rules of how to do this for pure states, show that the resulting $de$-dimensional mixed state has density matrix $\rho \otimes \rho'$, the Kronecker product of $\rho$ and $\rho'$.

Ans:

(a) Let $(i, j)$ be element from $[d] \times [e]$.

Then $(p \otimes q)_{i,j} = p_i p_j$

$$= Pr_p[i] Pr_q[j]$$

which is the probability of independently selecting $i$ from $p$ and $j$ from $q$.

(b) Given:

$$\rho = \sum_{i=1}^{d} p_i |\psi_i\rangle \langle \psi_i|$$

$$\rho' = \sum_{j=1}^{e} q_j |\phi_j\rangle \langle \phi_j|$$
We can write the density matrix for the combined system as:

\[
\sum_{i=1}^{d} \sum_{j=1}^{e} p_i q_j (|\psi_i\rangle \otimes |\phi_j\rangle)(|\psi_i\rangle \otimes \langle \phi_j|) = \sum_{i=1}^{d} \sum_{j=1}^{e} p_i q_j (|\psi_i\rangle \langle \psi_i|) \otimes (|\phi_j\rangle \langle \phi_j|)
\]

\[
= \sum_{i=1}^{d} \sum_{j=1}^{e} (p_i |\psi_i\rangle \langle \psi_i|) \otimes (q_j |\phi_j\rangle \langle \phi_j|)
\]

\[
= (\sum_{i=1}^{d} p_i |\psi_i\rangle \langle \psi_i|) \otimes (\sum_{j=1}^{e} q_j |\phi_j\rangle \langle \phi_j|)
\]

\[
= \rho \otimes \rho'
\]
5. Let $\rho \in \mathbb{C}^{d\times d}$ be a density matrix. Recall that for an observable (i.e., Hermitian matrix) $X \in \mathbb{C}^{d\times d}$, we define

$$E_\rho[X] = \langle \rho, X \rangle = \text{tr}(\rho^\dagger X) = \text{tr}(\rho X) = \sum_{i,j} \rho_{ij} X_{ij}. \quad (1)$$

(a) Prove that $E_\rho[I_{d\times d}] = 1$.

(b) Let $X, Y \in \mathbb{C}^{d\times d}$ be Hermitian and let $\alpha, \beta \in \mathbb{C}$. Prove “linearity of expectation”: $E_\rho[\alpha X + \beta Y] = \alpha E_\rho[X] + \beta E_\rho[Y]$. Also, show that $\alpha X + \beta Y$ is Hermitian if $\alpha, \beta \in \mathbb{R}$ (otherwise, we can’t be sure).

(c) Prove that $E_\rho[A^\dagger A] \geq 0$ for any matrix $A \in \mathbb{C}^{d\times d}$.

(d) Let $\rho' \in \mathbb{C}^{d\times d}$. Referring to Problem 4, prove that $E_{\rho \otimes \rho'}[X \otimes Y] = E_\rho[X]E_{\rho'}[Y]$. (This generalizes the classical probability fact that if $X$ and $Y$ are independent random variables then $E[XY] = E[X]E[Y]$.)

Ans:

(a)

$$E_\rho[I_{d\times d}] = \sum_{i,j} \rho_{ij} I_{ij} = \text{tr}(\rho I) = \text{tr}(\rho) = 1$$

(b) Prove "linearity of expectation“:

$$E_\rho[\alpha X + \beta Y] = \text{tr}(\rho(\alpha X + \beta Y))$$

$$= \text{tr}(\alpha \rho X + \beta \rho Y)$$

$$= \alpha \text{tr}(\rho X) + \beta \text{tr}(\rho Y)$$

$$= \alpha E_\rho[X] + \beta E_\rho[Y]$$

Next, show that $\alpha X + \beta Y$ is Hermitian:

$$\begin{align*}
(\alpha X + \beta Y)_{ij} &= \alpha X_{ij} + \beta Y_{ij} \\
&= \alpha X^*_{ji} + \beta Y^*_{ji} \\
&= (\alpha X_{ji} + \beta Y_{ji})^* \\
&= (\alpha X + \beta Y)^*_{ji}
\end{align*}$$
(c)

\[ \mathbb{E}_\rho[A^\dagger A] = \sum_{i,j} p_{ij} (A^\dagger A)_{ij} \]

\[ = \sum_{i,j} p_{ij} \sum_{i,j} A^\dagger_{ik} A_{kj} \]

\[ = \sum_{i,j} p_{ij} \sum_{i,j} A^*_{ki} A_{kj} \]

\[ = \sum_k \langle A_k | \rho | A_k \rangle \]

Since \( \rho \) is positive semidefinite (property of density matrix), then \( \sum_k \langle A_k | \rho | A_k \rangle \geq 0 \). Thus, \( \mathbb{E}_\rho[A^\dagger A] \geq 0 \).

(d)

\[ \mathbb{E}_{\rho \otimes \rho'}[X \otimes Y] = \text{tr}((\rho \otimes \rho')(X \otimes Y)) \]

\[ = \text{tr}((\rho X) \otimes (\rho' Y)) \]

\[ = \text{tr}(\rho X)\text{tr}(\rho' Y) \]

\[ = \mathbb{E}_\rho[X] \mathbb{E}_{\rho'}[Y] \]