Robust Multipath Routing in Large Wireless Networks

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Abstract—One of the challenges of wireless networks is to provide a reliable end-to-end path between two end hosts in the face of link and node outages. These can occur due to fluctuations in channel quality, node movement, or node failure. One mechanism that has been proposed is based on multipath routing, the idea being to establish two or more paths between the end hosts so that they always have a path between them with high probability in the face of outages. This naturally raises the question of how to discover these paths in an unknown, random wireless network to enable robust multipath routing. In order to answer this question, we model a random wireless network as a 2D spatial Poisson process. Based on the results of percolation highways in Franceschetti, et al. [1], we present accurate conditions that enable robust multipath routing. If the number of hops of a path between the end hosts is \( n \), then there exists a path between them in a strip of width proportional to \( \log n \). More precisely, there exist \( C \log n \) disjoint paths in a strip of width \( a(C, p) \cdot \log n \), where \( p \) is the probability that characterizes the availability of an individual wireless communication link. We derive tight bounds for the function \( a(C, p) \). This provides a useful guideline for the establishment of multiple paths in a real wireless network, namely that the width should grow logarithmically in the number of hops on the path between the hosts.

I. INTRODUCTION

Wireless networks are characterized by time-varying link characteristics and network topology. In such an environment, the network must accommodate these changes to provide end-to-end packet delivery in a seamless manner. To that end, proposals have been made to transfer data over multiple paths from a source to a destination. The presence of multiple paths allows an end-to-end connection to tolerate link failures and/or disappearances. Many multipath routing algorithms have been proposed; see [2] for a survey of multipath routing. Among the most intriguing are braided routing algorithms, which promise to provide robustness to failures and traffic changes at minimal overhead. In the context of this paper, a braided routing algorithm constructs a braid, which consists of a set of nodes and links that includes one or more paths between the source and destination. The idea is to construct a braid such that there is a good performing path between the source and destination even if a small number of links or nodes fail, or traffic on links/nodes in the braid changes dramatically.

The concept of a braid was first proposed in [3] where the intent was to generate a braid that would tolerate a single link failure. Braided routing has since been studied in a number of papers. Most relevant to our effort is the work of Manfredi, et al. [4], which introduced the notion of a \( k \)-hop braid. The idea here was to first construct a shortest path between source and destination, and then include all nodes within a distance \( k \) of the shortest path. They show through simulation that \( k \) need not be large in order to provide robustness against either node or link loss.

This paper studies how to enable robust multipath (or braided) routing. We consider a wireless network as a 2D spatial Poisson process, and provide the conditions and mechanism to establish the required multiple paths between end hosts. In summary, we establish the following:

1) Suppose the number of hops of a path between the end hosts is \( n \). In order to ensure robustness, the end hosts need to use all possible paths between them that lie in a strip of width at least proportional to \( \log n \).
2) If the strip width is at least \( a^*(C, p) \cdot \log n \) for a constant \( a^*(C, p) \), then there exist at least \( C \log n \) reliable disjoint paths across the strip, where \( p \) is the probability that characterizes the availability of an individual wireless communication link. We provide lower and upper bounds for \( a^*(C, p) \).

Our work relies on the results from Franceschetti, et al. [1] about the construction of “percolation highways” in a spatial Poisson process, which are based on prior results from percolation theory [5]. What distinguishes our work from [1] is that [1] only establishes a sufficient condition for the existence of multiple reliable paths (called percolation highways) for the asymptotic analysis in large networks, whereas we derive tighter conditions for the constant \( a^*(C, p) \) and evaluate the effectiveness of this approach with respect to finite networks.

II. MODEL AND PRELIMINARIES

A convenient way to model random outages is to consider a 2D Poisson point process\(^1\) of normalized unit density on a rectangular strip of size \( x \times y(x) \) between source \( s \) and destination \( d \) (see Fig. 1 (a)). Two nodes can communicate with each other reliably, if they are within a distance \( r \). The random locations of nodes in a Poisson point process capture

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\(^{1}\)One can consider an alternative point process where \( n \) nodes are placed on the plane by uniform distribution. But this point process converges to a Poisson point process asymptotically.
uncertain node positions. Also, to capture node failure, we let each node be present independently with probability \( p_n \). Note that the induced point process is identical to a Poisson point process of density \( p_n \). We seek to determine the strip width \( y(x) \), such that at least one left-right path exists across the strip. Namely, \( s \) and \( d \) need to maintain reliable communication not more than a range \( y(x) \) at the ends for enabling robust communication in the face of outages at the nodes among them.

A natural approach to address the connectivity of a Poisson point process is to map it to a 2-D lattice network, and use the results from bond percolation theory [5]. The mapping is based on the tilted cells of length \( c \), where each cell represents a possible link in either vertical or horizontal orientations (as shown as dotted links in Fig. 1 (b)). All these links form an \( n \times m(n) \) lattice network, where \( n = \left\lceil \frac{y(x)}{2c} \right\rceil \). We enable a link in a cell, if and only if there is at least one node in the cell. Suppose \( r \geq 2\sqrt{2}c \). Hence, if there is a left-right path in the lattice network, then there is a left-right path in the 2-D Poisson process with neighboring nodes within a distance \( r \) of each other. In the lattice network, the presence of links is characterized by a probability \( p \). We set

\[
p = 1 - e^{-p_n c^2}
\]

which represents the presence of a node in a cell in the Poisson point process. Probability \( p \) captures the availability of an individual wireless communication link.

The central question of setting the proper strip width \( y(x) \) reduces to setting the width \( m(n) \) in the induced lattice network. Furthermore, we not only study the undirected left-to-right paths, but also the directed left-to-right paths, which are formed by consecutive directed left-to-right links (see the green path in Fig. 1 (d)). The presence of such paths allows a simple discovery mechanism that does not traverse nodes in the opposite horizontal direction. Therefore, we also map a Poisson point process to a semi-directed lattice network, as in Fig. 1 (e) with undirected vertical links and directed horizontal links. Thus, if there is a directed left-to-right path in the semi-directed lattice network, then there is a directed left-to-right path in the 2-D Poisson process (see Fig. 1 (f)).

A. Lattice Network

We label the lattice network by \( [0, n] \times [-\frac{m(n)}{2}, \frac{m(n)}{2}] \). Supposing that a link is present with probability \( p \) independently over other links. It has been shown [5] that there exists a percolation threshold, \( p_c = \frac{1}{2} \) such that whenever \( p > p_c \), the system percolates. In other words there exists a single set of network nodes of infinite size that are connected by the randomly present edges. This result also applies to a Poisson point process that is mapped to a lattice network.

We are particularly interested in determining the value of \( m(n) \) such that, with high probability there is a rich set of left-to-right paths traversing the strip accessible to the source and destination. Formally, we denote \( R_{n,m(n)} \) as an \( n \times m(n) \) lattice network, and \( \{ \leftrightarrow R_{n,m(n)} \} \) as the event that there exists a left-to-right path in \( R_{n,m(n)} \). Based on [5, Sec. 11.5], we have percolation within the lattice network, and hence \( \mathbb{P}\{ \leftrightarrow R_{n,m(n)} \} \to 1 \), as \( n \to \infty \) provided that \( p > p_c \) and \( m(n) = \frac{\log n}{a} \) for \( a > a^\ast(p) \) where \( a^\ast(p) \) is a decreasing function of \( p \). Moreover, [1] has shown that when there is percolation, there are also \( \Theta(\log n) \) disjoint left-to-right paths.

The above carries the following interpretation. Given a link up probability \( p > p_c \), there exists a constant depending on \( p \), \( a^\ast(p) \) such that if we choose a strip of width \( a \log n \) for \( a > a^\ast(p) \), with high probability we will have at least one left-right path (in actuality \( \Theta(\log n) \) paths). Moreover, there also exists a constant depending on \( p \) and \( C \), \( a^\ast(C,p) \) such that if we choose a strip of width \( a \log n \) for \( a > a^\ast(C,p) \), with high probability we will have at least \( C \log n \) left-right paths. We are motivated to study the values of \( a^\ast(p) \) and \( a^\ast(C,p) \).

III. Simulation Studies

In this section we motivate our theoretical work with a series of simulation studies. We study the properties of random paths in four cases: 1) an undirected lattice network as in Fig. 1 (b), 2) a semi-directed lattice network as in Fig. 1(e), 3) a 2-D
Poison point process with undirected links, 4) a 2-D Poisson point process with directed left-to-right links as in Fig. 1(d).

In undirected and semi-directed lattice networks, we first estimate the probability of a left-to-right path with varying lengths, widths and edge up probability, \( p \) in Figs. 2 and 5 respectively. Each horizontal panel relates to a fixed value of \( p \) with values ranging from \( p = 0.75 \) to \( p = 0.95 \) and for values of \( n \) ranging from \( n = 10 \) to \( n = 1000 \).

We are interested in how the width of the strip affects the probability of a left-to-right path in a randomly constructed sub-graph of the lattice network. In order to gain some insight into this dependence we illustrate the form of \( m(n) \) that yields a fixed probability of establishing a path. Figs. 2 and 5 show the contour plots of the probability of a path from the source to the destination node as the length and width of the lattice network is varied. The \( x \)-axis is shown in a log scale and we observe roughly linear contour lines of fixed probability which supports the finding that widths of strips scale in the log of their length. For each contour plot we set \( p = 0.75 \).

Figs. 4 and 7 show the results of solving a maximum flow problem for a randomly constructed lattice network with edge up probability, \( p \), and for several values of the length, \( n \), ranging over three orders of magnitude. The edge capacity of each edge was set to one. Super nodes were used as the source and destination of flow. The source super node was connected by an edge of infinite capacity to the destination super node. Maximizing the flow from the source (super) node to the destination (super) node thus determines the number of edge disjoint left-to-right paths of the lattice network.

Additionally, Figs. 3 and 6 show a simple approximation to the mean number of edge disjoint paths based on the minimum of a set of Binomial random variables. We can see that the approximation becomes more accurate with larger values of \( p \). The approximation is shown in the dashed lines whereas the solid lines show the solution to the maximum flow problem.

The Binomial approximation is obtained by restricting the
selection of minimal cuts to those which are vertical. Each vertical cut has a capacity which is a Binomial random variable with parameters $\text{Bin}(m(n), p)$. The least such vertical cut is thus an upper bound on the maximum flow problem where non-vertical cuts are neglected. This upper bound is the minimum of $n$ independent such Binomial random variables corresponding to the $n$ possible vertical cuts separating the left and right edges. It is then straightforward to evaluate the mean of $X_{\text{min}}$, the minimum of $n$ independent, identically distributed Binomial random variables, $X_i \sim \text{Bin}(m(n), p)$, using the expression (see [6, page 97]):

$$\mathbb{E}(X_{\text{min}}) = \sum_{x=0}^{m(n)-1} (1 - P(X_i \leq x))^n \quad (2)$$

Next, we repeat a similar study to Poisson point processes with undirected and directed left-to-right links. The results are shown in Figs. 8-12. Here, we vary the values of communication range $r$. We note that Eqn. (1) and $r \geq 2\sqrt{2}c$, we can relate $r = 2.5$ to $p = 0.54$ and $r = 2$ to $p = 0.39$. We observe that the connectivity in Poisson point process is greater than that in the corresponding lattice network.

### A. Key Observations and Implications

From the simulation results, we obtain a number of key observations and implications:

1) For both lattice networks and Poisson point processes, simulation results for the probability of a left-to-right path, and the number of disjoint left-to-right paths exhibit a similar pattern. Hence, the connectivity results in lattice networks can serve as lower bounds for those in Poisson point processes.

2) The restriction to semi-directed lattice network produces results that differ little from those for an undirected lattice network. This suggests that the left-to-right paths in a lattice network, when they exist, are mostly paths that need not traverse in the reverse horizontal direction.

3) There appears to be a linear relationship between the width of the strip and the length in log scale for the same probability of a left-to-right path. This corroborate with the analytical results in percolation theory.

In the following, we provide analytical results to corroborates the relation between width and the length of the strip.
IV. Upper Bounds

In this section, we provide upper bounds for \( a^*(p) \) and \( a^*(C, p) \), based on prior results [1], [5] on the left-to-right disjoint paths in an \( n \times a \log n \) lattice network, as \( n \to \infty \).

**Theorem 1:** If \( p > \frac{2}{3} \), we obtain:
\[
 a^*(p) < \frac{-1}{\log(3(1-p))} \tag{3}
\]

**Theorem 2:** If \( p > \frac{2}{3} \), we obtain:
\[
 a^*(C, p) < \frac{-1 - C \log \left( \frac{p}{0.15(p-\frac{2}{3})} \right)}{\log(3(1-p) + 0.15(p-\frac{2}{3}))} \tag{4}
\]

The proofs are omitted due to space constraints. We illustrate the upper bounds for \( a^*(C, p) \) in Fig. 14. The basic idea is that we first denote \( B_m \) to be a square lattice of side length \( m \) embedded in the 2-D lattice. We assume \( B_m \) is centered at the origin of the 2-D lattice. Let \( \{0 \leftrightarrow \partial B_m\} \) be the event that there exists a path connecting the origin and the boundary of \( B_m \). We can obtain \( \mathbb{P}\{0 \leftrightarrow \partial B_m\} \) from percolation theory. Then, we can study \( \mathbb{P}\{\leftrightarrow R_{n,a \log n}\} \), the probability that there exists a left-to-right path in \( R_{n,a \log n} \), and \( a^*(C, p) \).

Fig. 14. Plot of the upper bounds for \( a^*(C, p) \).

V. Lower Bound

In this section, we establish a lower bound for \( a^*(p) \), based on the number of left-to-right disjoint paths in an \( n \times a \log n \) lattice network, as \( n \to \infty \).

Consider the strip \( [0, n] \times [-\frac{m(n)}{2}, \frac{m(n)}{2}] \). Divide the strip into \( n \) columns where column \( i \) consists of the edges \( e = ((i-1, j), (i, j)), j = \frac{m(n)}{2}, \ldots, \frac{m(n)}{2} \). Let \( X_i \) denote the number of vertical links that are up for column \( i \). Note that \( \{X_i\}_{i=1}^n \) are independent, identically distributed Binomial random variables, \( X_i \sim \text{Bin}(m(n), p) \). Also note that at most \( X_i \) links can carry flow in the \( i \)-th column. Similarly at most \( X_i \) nodes can be carrying flow on either side of this column of links. Therefore, the maximum flow that can be carried is upper bounded by
\[
 X_{\leftrightarrow} \leq X_{\min} \triangleq \min_{i=1,\ldots,n} X_i \tag{5}
\]
Hence, as \( n \to \infty \),
\[
 \mathbb{P}\{\leftrightarrow R_{n,a \log n}\} = \mathbb{P}\{X_{\leftrightarrow} \neq 0\} \to 1 \Rightarrow \mathbb{P}\{X_{\min} \neq 0\} \to 1 \tag{6}
\]
Since \( X_i \) is a Binomial random variable, we obtain:
\[
 \mathbb{P}\{X_{\min} \neq 0\} = (1 - (1 - p)^{a \log n})^n \tag{7}
\]
By \( 1 - q \leq \left(\frac{1}{e}\right)^q \) for \( 0 < q < 1 \), we obtain:
\[
 \mathbb{P}\{X_{\min} \neq 0\} \leq \left(\frac{1}{e}\right)^n (1 - p)^{a \log n} = \left(\frac{1}{e}\right)^n \cdot n \cdot (1 - p)^{a \log(1-p)} \tag{8}
\]
Hence,
\[
 \lim_{n \to \infty} \mathbb{P}\{X_{\leftrightarrow} \neq 0\} = 1 \Rightarrow \lim_{n \to \infty} \left(\frac{1}{e}\right)^n \cdot n \cdot (1 - p)^{a \log(1-p)} \geq 1 \Rightarrow \lim_{n \to \infty} n \cdot (1 - p)^{a \log(1-p)} = 0 \Rightarrow a \log(1-p) < -1 \tag{9}
\]

Therefore, considering \( \log(1-p) < 0 \), we obtain the following lower bound for \( a^*(p) \).

**Theorem 3:** We obtain:
\[
 a^*(p) > \frac{-1}{\log(1-p)} \tag{10}
\]

We illustrate the upper and lower bounds for \( a^*(p) \) in Fig. 15. We also plot the estimated values of \( a^*(p) \) from the simulation studies in Sec. III. For instance, \( a^*(0.75) \approx \frac{2.3}{3} \log_{10} e = 1.05 \) from the slopes of the contours in Fig. 3. We observe that our theoretical results are consistent with the simulation studies, as they are between the upper and lower bounds.

![Fig. 15. Plot of the upper and lower bounds for \( a^*(p) \). The red points are the estimated values of \( a^*(p) \) from the simulation studies in Sec. III.](image)

VI. Conclusion

This paper focused on the problem of identifying how wide a strip ought to be in a wireless network modeled by a 2-D Poisson point process, so as to ensure an end-to-end path between a source and destination with high probability. We mapped this problem into that of identifying conditions under which percolation occurs in a 2-D lattice network. Drawing on results on 2-D lattice networks, we showed that the width of a strip had to be at least \( a \log n \) where \( n \) is the distance in hops between source and destination. Furthermore, we identified bounds on the value of \( a \) as a function of the probability that characterizes the availability of an individual wireless communication link. We also observe from our simulation on semi-directed lattices that the connectivity properties of paths with restricted direction have a little difference than those with unrestricted direction.

REFERENCES